Sommese Vanishing for non-compact manifolds

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Introduction

Let X be a smooth algebraic manifold of dimension $\operatorname{dim} X = n$ over an algebraically closed field k of characteristic $\operatorname{char} k = 0$. Assume that X is equipped with a birational projective map $\pi : X \to Y$ to a normal irreducible algebraic manifold Y. The Kodaira Vanishing Theorem admits a well-known generalization to this relative situation, namely, the Grauert-Riemenschneider Vanishing Theorem, which claims that the higher direct image sheaves $R^k\pi_*\Omega^n_X$ on Y are trivial for $k > 0$. One could conjecture that the stronger Kodaira-Nakano Theorem can also be generalized to the relative situation, so that one would have $R^p\pi_*\Omega^q_X = 0$ for $p + q > n$. However, a moment's reflection – take X to be the blowup of a smooth point – shows that this is not true, unless one imposes some additonal assumptions on X or on the map π.

The goal of this paper is to prove the relative version of Kodaira-Nakano Theorem for one set of such additional assumptions. We will prove (see Theorem 1.1 below) that $R^p\pi_*\Omega^q_X = 0$ if $p + q > \operatorname{dim} X \times_Y X$, the dimension of the fibered product of X with itself over Y. We do not need the map $X \to Y$ to be birational, projective is enough. In the case when $\operatorname{dim} X \times_Y X = X$, the condition on degrees is the original one, that is, $p + q > n$. Such maps $X \to Y$ are necessarily generically finite, and they are known as *semismall maps.*

A vanishing theorem of the same type has been proved some time ago by A. Sommese [S], but only in the case when the base manifold Y is compact. Sommese Theorem is global – rather than consider the direct images, he considers the global cohomology groups $H^p(X, \Omega^q_X \otimes \pi^*M)$, where M is an ample line bundle on Y. There is a well-known procedure for deducing the vanishing of higher direct images from such a global vanishing, but there is a hitch – in order to apply it, one has to compactify X, Y and $\pi : X \to Y$ so that all assumptions on π are preserved. Unfortunately, usually it is not possible to compactify a semismall map so that it stays semismall.

Our proof uses essentially the same ideas, but we arrange the induction on dimension in a slightly different way; this allows us to use an arbitrary smooth compactification but make all the unpleasantness stay at infinity, where it belongs. The characteristic 0 assumption is used in two places: firstly, we need it to use Hodge Theory, secondly, we need the resolution of singularities

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to construct smooth compactifications. The Hodge Theory part is not critical, since it can be done in characteristic p by the famous method of Delign-Illusie [DI] (at the cost of some additional assumptions on X such as liftability to the ring of second Witt vectors). But the resolution of singularities seems essential.

As an application, we prove a general theorem on the topology of symplectic contractions over $\mathbb{C}$ (Theorem 3.2). Roughly speaking, we assume that Y is affine, $\pi : X \to Y$ is birational, and X carries a non-degenerate closed 2-form Ω, and we deduce that for the fiber $E_y = \pi^{-1}(y) \subset X$ of the map $\pi : X \to Y$ over any closed point $y \in Y$, the cohomology group $H^k(E_y, \mathbb{C})$ is trivial if $k = 2p+1$, and carries a pure Hodge structure of type (p,p) if $k = 2p$. We note that in the particular case of the so-called *Springer resolution*, this statement has been proved by a direct geometric argument in [dCLP].

The author is definitely not an expert in the field of vanishing theorems. For those like him, he would like to mention at this point that there is an excellent and standard reference for all things related to algebraic vanishing theorems, namely, the book [EV] by H. Esnault and E. Viehweg. In particular, it is from this book that the author has learned the Sommese Theorem, as well as its proof and all the ideas needed for the proof. The present paper has also been very much influenced by the beautiful paper [dCM] devoted to the geometry of semismall maps.

The fibered product $X \times_Y X$ appears in the statements for purely numerical reasons. It would be interesting to see if it has any real singificance.

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1 Statements and preliminaries.

Fix an algebraically closed field k of characteristic $\operatorname{\mathsf{char}} k = 0$. Let X be a regular scheme over k of dimension $\operatorname{\mathsf{dim}} X = n$. Assume given a projective map $\pi : X \to Y$ from X to a normal irreducible algebraic variety Y over k. Denote by $X \times_Y X$ the fibered product of X with itself over Y. Here is our main theorem.

Theorem 1.1. *For any $p, q \geq 0$ such that $p + q > \dim X \times_Y X$ we have*

$$R^p \pi_* \Omega^q_X = 0.$$

Before we start proving it, we need to set up some preliminary machinery. First of all, we explain the numerical bound in Theorem 1.1.

Lemma 1.2. *Assume given a smooth variety X over k equipped with a proper map $\pi : X \to Y$ into an affine variety Y. Then*

$$H^l_{DR}(X) = 0$$

whenever $l > \dim X \times_Y X$.

Proof. Since k has characteristic 0, we may assume $k = \mathbb{C}$ and work in the analytic topology. We can compute the cohomology $H^\bullet_{DR}(X) = H^\bullet(X, \mathbb{C})$ by applying the Leray spectral sequence for the map $\pi : X \to Y$. Let $Y_p \subset Y$ be the closed subvariety of points $y \in Y$ such that $\dim \pi^{-1}(y) \geq p$. By proper base change, the sheaf $R^k\pi_*\mathbb{C}$ is supported on Y_p, where p is the smallest integer such that $k \leq 2p$. Since $Y_p \subset Y$ are affine varieties, the group

$$H^{l-k}(Y, R^k\pi_*\mathbb{C}) = H^{l-k}(Y_p, R^k\pi_*\mathbb{C})$$

vanishes whenever $l - k > \dim Y_p$. On the other hand, $\dim X \times_Y X \geq \dim Y_p + 2p$, so that $l > \dim X \times_Y X$ implies $l > \dim Y_p + 2p$. Collecting all this together, we see that

$$H^{l-k}(Y, R^k\pi_*\mathbb{C}) = 0$$

whenever $l > \dim X \times_Y X$. □

Next, assume given a smooth manifold X over k equipped with a line bundle L.

Definition 1.3. A *flag* for L is a sequence

$$\emptyset = W_l \subset \cdots \subset W_1 \subset W_0 = X$$

such that W_{i+1} is either empty or a Cartier divisor in W_i, and the line bundle $\mathcal{O}_{W_i}(W_{i+1})$ on W_i is isomorphic to the restriction of the line bundle L onto W_i. A flag is called *smooth* if all the W_i are smooth.

If in addition X is equipped with a simple normal crossing divisor D, then a flag W_i is said to be *transversal to D* if every W_i intersects transversally with every intersection of the components of the divisor D. If this is the case, then in particular the union $D \cup W_1$ is a simple normal crossing divisor in X. Moreover, by induction $(D \cap W_i) \cup W_{i+1}$ is a normal crossing divisor in W_i for every i.

Lemma 1.4. *Assume given a line bundle L on a smooth manifold X over k. If L is base-point free, it admits a smooth flag. Moreover, if X is equipped with a normal crossing divisor D, then such a flag can be chosen so that it is transversal to D.*

Proof. By induction on $\dim X$, it suffices to find a smooth divisor $W_1 \subset X$ in the linear system $|L|$, transversal to D if necessary. By the Bertini Theorem, a generic member of $|L|$ will do. $\square$

Remark 1.5. If the line bundle L is very ample, a flag for L is of length $n = \dim X$. However, this need not be so if L is only base-point free. In the extreme case of trivial L, a flag consists of two subvarieties: $W_0 = X$ and $W_1 = \emptyset$. Nevertheless, it is a smooth flag.

Lemma 1.6. *Let X be a smooth manifold over k equipped with a normal crossing divisor D, a line bundle L and a smooth flag W_i for L which is transversal to D. Then the logarithmic de Rham sheaves $\Omega^\bullet_X\langle D\rangle \otimes L$ admit an increasing filtration F_i such that*

$$\mathsf{gr}^i_F\,(\Omega^\bullet_X\langle D\rangle \otimes L) = \Omega^\bullet_{W_i}\langle (D \cap W_i) \cup W_{i+1}\rangle.$$

Proof. By induction on $\dim X$, it suffices to prove that the natural restriction map $\Omega^\bullet_X \otimes L \to \Omega^\bullet_{W_1} \otimes L$ and the embedding $\Omega^\bullet\langle D \cup W_1\rangle \subset \Omega^\bullet\langle D\rangle \otimes L$ induced by the embedding $\mathcal{O} \subset \mathcal{O}(W_1) \cong L$ together give a short exact sequence

$$0 \;\to\; \Omega^\bullet_X\langle D \cup W_0\rangle \;\to\; \Omega^\bullet_X\langle D\rangle \otimes L \;\to\; \Omega^\bullet_{W_1}\langle D \cap W_1\rangle \otimes L \;\to\; 0. \tag{1.1}$$

This sequence is well-known, see e.g. [EV, Property 2.3c]. We give a proof for the sake of completeness. The claim is local, so that we may assume that the manifold X is $\mathbb{A}^n$ with coordinates $z_1, \dots, z_n$, that $W_1 \subset X$ is the coordinate hyperplane $z_1 = 0$, and that $D \subset X$ is the union of the coordinate hyperplanes $z_i = 0$, $i = 2, \dots, l$ for some $l \leq n$. We have

$$\Omega^\bullet_X\langle D \cup W_1\rangle = \Omega^\bullet_{\mathbb{A}^1}\langle o\rangle \boxtimes (\Omega^\bullet_{\mathbb{A}^1}\langle o\rangle \boxtimes \cdots \boxtimes \Omega^\bullet_{\mathbb{A}^1}\langle o\rangle) \boxtimes (\Omega^\bullet_{\mathbb{A}^1} \boxtimes \cdots \boxtimes \Omega^\bullet_{\mathbb{A}^1})\,,$$

where $\Omega^\bullet_{\mathbb{A}^1}\langle o\rangle$ is the logarithmic de Rham complex of $\mathbb{A}^1$ with singularities at the origin point $o \in \mathbb{A}^1$, and the logarithmic factors correspond to coordinates z_1 and $z_2, \dots, z_l$. Moreover, we have

$$\Omega^\bullet_X\langle D\rangle \otimes L = (\Omega^\bullet_{\mathbb{A}^1} \otimes \mathcal{O}(o)) \boxtimes (\Omega^\bullet_{\mathbb{A}^1}\langle o\rangle \boxtimes \cdots \boxtimes \Omega^\bullet_{\mathbb{A}^1}\langle o\rangle) \boxtimes (\Omega^\bullet_{\mathbb{A}^1} \boxtimes \cdots \boxtimes \Omega^\bullet_{\mathbb{A}^1})\,,$$

and

$$\Omega^\bullet_{W_1}\langle (D \cap W_1)\rangle \otimes L = k_o \boxtimes (\Omega^\bullet_{\mathbb{A}^1}\langle o\rangle \boxtimes \cdots \boxtimes \Omega^\bullet_{\mathbb{A}^1}\langle o\rangle) \boxtimes (\Omega^\bullet_{\mathbb{A}^1} \boxtimes \cdots \boxtimes \Omega^\bullet_{\mathbb{A}^1})\,,$$

where k_o is the skyscraper sheaf supported at the origin. Therefore (1.1) follows from the obvious exact sequence

$$0 \longrightarrow \Omega^\bullet_{\mathbb{A}^1}\langle o\rangle \longrightarrow \Omega^\bullet_{\mathbb{A}^1} \otimes \mathcal{O}(o) \longrightarrow k_o \longrightarrow 0.$$

This finishes the proof. $\square$

2 Proofs.

We can now start proving Theorem 1.1. First we reduce the problem to a particular case.

Lemma 2.1. *To prove Theorem 1.1, it suffices to prove that for every p, q with $p+q > \mathsf{dim}\, X \times_Y X$, the sheaf*

$$R^p \pi_* \Omega^q_X$$

vanishes if it is supported set-theoretically on a finite union $Y_0 \subset Y$ of closed points in Y.

Proof. Apply induction on $n = \mathsf{dim}\, X$. Assume that Theorem 1.1 is proved for all X' with $\mathsf{dim}\, X' < n$, and assume that support of the sheaf $R^p \pi_* \Omega^q_X$ has an irreducible component $Y_0 \subset Y$ of dimension $r = \mathsf{dim}\, Y_0 > 0$. Shrinking Y if necessary, we may assume that Y_0 is smooth and affine, and that Y admits a smooth projection $\rho_0 : Y \to \mathbb{A}^r$ into the affine space $\mathbb{A}^r$ which restricts to an étale map $\rho_0 : Y_0 \to U$ from $Y_0 \subset Y$ onto an open subset $U \subset \mathbb{A}^r$. Shrinking Y even further, we can also assume that the associated projection $\rho = \rho_0 \circ \pi : X \to Y_0$ is a smooth map. Moreover, we may assume that $R^p \pi_* \Omega^q_X$ is supported on $Y_0 \subset Y$. The smooth map $\rho : X \to U$ induces a filtration on Ω^q_X with associated graded pieces

$$\Omega^k(X/U) \otimes \rho^* \Omega^l_U, \qquad k + l = q, \tag{2.1}$$

where $\Omega^k(X/U)$ is the sheaf of relative k-forms on X/U. The projection formula gives

$$R^p \pi_* \left(\Omega^k(X/U) \otimes \rho^* \Omega^l_U \right) = R^p \pi_* \Omega^k_X \otimes \Omega^l_U.$$

This sheaf tautologically vanishes whenever $l > r$. For any closed point $u \in U$, the fiber $X_u = \rho^{-1}(u) \subset X$ is smooth and equipped with a projective map $\pi : X_u \to Y_u$ into the fiber $Y_u = \rho_0^{-1}(u) \subset Y$. By induction on $n = \mathsf{dim}\, X$, we may assume that

$$R^p \pi_* \Omega^k_{X_u} = 0$$

whenever $p + k > \mathsf{dim}\, X_u \times_{Y_u} X_u$. By base change, this implies that

$$R^p \pi_* \Omega^k(X/U) = 0$$

if $p + k > \mathsf{dim}\, X_u \times_{Y_u} X_u$ for all $u \in U$. By our assumptions, for a generic $u \in U$ we have

$$\mathsf{dim}\, X \times_Y X = \mathsf{dim}\, Y_0 + \mathsf{dim}\, X_u \times_{Y_u} X_u.$$

Shrinking U if necessary, we may assume that this holds for any closed point $u \in U$. Therefore $p+q > \mathsf{dim}\, X \times_Y X$ and $l \leq \mathsf{dim}\, Y_0$ implies $k + p = p + q - l > \mathsf{dim}\, X_u \times_{Y_u} X_u$. We conclude that

$$R^p \pi_* \left(\Omega^k(X/Y_0) \otimes \rho^* \Omega^l_U \right) = 0$$

whenever $p+q = p+k+l > \mathsf{dim}\, X \times_Y X$. Applying the spectral sequence associated to the filtration (2.1), we conclude that indeed, $R^p\pi_*\Omega^q_X = 0$. $\square$

Proof of Theorem 1.1. By Lemma 2.1, we may assume that all the sheaves $R^p\pi_*\Omega^q_X$ with $p+q > \mathsf{dim}\, X \times_Y X$ are supported in a finite subset in Y. Without loss of generality we may also assume that Y is affine. Choose a projective variety $\overline{Y}$, a smooth projective variety $\overline{X}$ and a projective map $\pi : \overline{X} \to \overline{Y}$ so that $Y \subset \overline{Y}$ is a dense open subset in $\overline{Y}$, $X = \pi^{-1}(Y) \subset \overline{X}$ is a dense open subset in $\overline{X}$, $\pi : \overline{X} \to \overline{Y}$ extends the given map $\pi : X \to Y$, and the complement $D = \overline{X} \setminus X$ is a simple normal crossing divisor in $\overline{X}$. Choose an ample line bundle M on $\overline{Y}$. Let $l >> 0$ be an integer large enough so that the sheaves

$$R^p\pi_*\Omega^q_{\overline{X}} \otimes M^{\otimes l}$$

are acyclic for all p, q. Replace M with $M^{\otimes l}$.

The line bundle $L = \pi^*M$ on $\overline{X}$ is base-point free. By Lemma 1.4, there exists a smooth flag W_i for the bundle L which is transversal to D. By construction, we have $W_i = \pi^{-1}(Z_i)$, where Z_i form a flag for M on $\overline{Y}$. In particular, for any i the complement $W_i \setminus W_{i+1}$ comes equipped with a projective map

$$\pi : W_i \setminus W_{i+1} \to Z_i \setminus Z_{i+1},$$

and the intersection $(Z_i \setminus Z_{i+1}) \cap Y$ is affine. By Lemma 1.2, we have

$$H^l_{DR}\left((W_i \setminus W_{i+1}) \cap X\right) = 0$$

whenever $l > \mathsf{dim}\, W_i \times_{Z_i} W_i$. If $l > \mathsf{dim}\, X \times_Y X$, then the vanishing holds for all i.

The cohomology $H^l_{DR}((W_i \setminus W_{i+1}) \cap X)$ can be computed by means of the logarithmic de Rham complex $\Omega^\bullet_{W_i}\langle (D \cap W_i) \cup W_{i+1}\rangle$. Moreover, the Hodge-de Rham spectral sequence for this complex degenerates, so that we have

$$H^p\left(\overline{X}, \Omega^q_{W_i}\langle (D \cap W_i) \cup W_{i+1}\rangle\right) = 0$$

whenever $l = p+q > X \times_Y X$. Applying Lemma 1.6, we deduce that

$$H^p\left(\overline{X}, \Omega^q_{\overline{X}}\langle D\rangle \otimes L\right) = 0 \tag{2.2}$$

under the same assumption on p, q.

Consider now the sheaves $\Omega^\bullet_{\overline{X}} \otimes L$ on $\overline{X}$ and the Leray spectral sequence for the map $\pi : \overline{X} \to \overline{Y}$. By our assumption on $L = \pi^*M$, the spectral sequence degenerates, and we have

$$H^p\left(\overline{X}, \Omega^q_{\overline{X}} \otimes L\right) = H^0\left(\overline{Y}, R^p\pi_*\Omega^q_{\overline{X}} \otimes M\right)$$

for every p, q. Since $Y \subset \overline{Y}$ is affine, the same degeneration holds over Y.

Assume from now on that $p+q > \mathsf{dim}\, X \times_Y X$. By our general assumption, the sheaf $R^p\pi_*\Omega^q_X$ is supported in a finite union of closed points in Y. In

particular, its support is distinct from the complement $\overline{Y} \setminus Y$, and it is therefore a direct summand in the sheaf $R^p\pi_*\Omega^q_{\overline{X}}$ on $\overline{Y}$. We conclude that the natural restriction map

$$(2.3) \quad H^p\left(\overline{X}, \Omega^q_{\overline{X}} \otimes L\right) = H^0(\overline{Y}, R^p\pi_*\Omega^q_{\overline{X}}) \longrightarrow \\ \longrightarrow H^p\left(X, \Omega^q_X \otimes L\right) = H^0(Y, R^p\pi_*\Omega^q_X)$$

is surjective.

On the other hand, denote by $j : X \to \overline{X}$ the open embedding. Since $D = \overline{X} \setminus X$ is a divisor in $\overline{X}$, the map j is affine, so that the higher direct images $R^k j_*\mathcal{E}$ are trivial for $k \geq 1$ and arbitrary coherent sheaf $\mathcal{E}$ on X. Therefore the restriction map (2.3) is induced by the embedding of sheaves $\Omega^q_{\overline{X}} \otimes L \subset j_*\Omega^q \otimes L$, which factors as

$$\Omega^q_{\overline{X}} \otimes L \longrightarrow \Omega^q_{\overline{X}}\langle D\rangle \otimes L \longrightarrow j_*\Omega^q_X \otimes L.$$

Since the sheaf in the middle has no cohomology in degree p by (2.2), the composition induces 0 on the cohomology groups of degree p. We conclude that

$$H^p(X, \Omega^q_X \otimes L) = H^0(R^p\pi_*\Omega^q_X \otimes M) = 0.$$

Since Y is affine, this proves the Theorem. $\square$

3 Applications.

We will now apply Theorem 1.1 to derive a topological vanishing theorem for symplectic manifolds. Assume that X is a smooth algebraic variety over k equipped with a non-degenerate closed 2-form $\Omega \in H^0(X, \Omega^2_X)$. Assume moreover that X is equipped with a projective birational map $\pi : X \to Y$ onto a normal algebraic variety Y.

Lemma 3.1. *The map $\pi : X \to Y$ is semismall, in other words,* $\mathsf{dim}\, X \times_Y X = \mathsf{dim}\, X$.

Proof. For any $p \geq 0$, let $Y_p \subset Y$ be the closed subvariety of points $y \in Y$ such that $\mathsf{dim}\, \pi^{-1}(y) \geq p$. It suffices to prove that $\mathsf{codim}\, Y_p \geq 2p$. By [K, Lemma 2.9], there exists an open dense subset $U \subset Y_p$ such that the restriction Ω_F of the form Ω onto the smooth part $F \subset \pi^{-1}(U)$ of the set-theoretic preimage $\pi^{-1}(U) \subset X$ satisfies

$$\Omega_F = \pi^*\Omega_U$$

for some 2-form $\Omega_U \in H^0(U, \Omega^2_U)$. Therefore the rank $\mathsf{rk}\, \Omega_F$ satisfies $\mathsf{rk}\, \Omega_F \leq \mathsf{dim}\, U$. On the other hand, since Ω is non-degenerate on X, we have $\mathsf{rk}\, \Omega_F \geq \mathsf{dim}\, F - \mathsf{codim}\, F$. Together these two inequalities give $\mathsf{codim}\, Y_p = \mathsf{codim}\, F + p \geq \mathsf{dim}\, F - \mathsf{dim}\, U + p = 2p$, as required. $\square$

Theorem 3.2. *Let $\pi : X \to Y$ be a projective birational map with smooth and symplectic X. Let $y \in Y$ be a closed point, and let $E_y = \pi^{-1}(y) \subset X$ be the set-theoretic fiber over the point y. Then for odd k we have $H^k(E_y, \mathbb{C}) = 0$, while for even $k = 2p$ the Hodge structure on $H^k(E_y, \mathbb{C})$ is pure of weight k and Hodge type (p,p).*

Lemma 3.3. *Let p be an integer, and let V be an $\mathbb{R}$-mixed Hodge structure with Hodge filtration $F^\bullet$ and weight filtration $W_\bullet$. Assume that $W_{2p}V = V$ and $F^pV = V$. Then V is a pure Hodge-Tate structure of weight $2p$ (in other words, every vector $v \in V$ is of Hodge type (p,p)).*

Proof. Since $V = F^pV$, the same is true for all associated graded pieces of the weight filtration on V. Therefore we may assume that V is pure of weight $k \leq 2p$. If $k < 2p$, we must have $V = F^pV \cap \overline{F^pV} = 0$, which implies $V = 0$. If $k = 2p$, the same equality gives $V = V^{p,p}$. $\square$

Proof of Theorem 3.2. By Lemma 3.1, Theorem 1.1 applies to $\pi : X \to Y$ and shows that

$$R^p\pi_*\Omega^q_X = 0 \tag{3.1}$$

whenever $p + q > \operatorname{dim} X$. Since X is symplectic, we have an isomorphism $\mathcal{T}_X \cong \Omega^1_X$ between the tangent and the cotangent bundle on X. This implies that $\Omega^q_X \cong \Omega^{\operatorname{dim} X - q}_X$, and (3.1) also holds whenever $p > q$.

Denote by $\mathfrak{X}$ the completion of the variety X in the closed subscheme $E_y = \pi^{-1}(y) \subset X$. Since the map $\pi : X \to Y$ is proper, by proper base change the group

$$H^p(\mathfrak{X}, \Omega^q_{\mathfrak{X}})$$

for any p, q coincides with the completion of the stalk of the sheaf $R^p\pi_*\Omega^q_X$ at the point $y \in Y$. Therefore $H^p(\mathfrak{X}, \Omega^q_{\mathfrak{X}}) = 0$ whenever $p > q$. The stupid filtration on the de Rham complex $\Omega^\bullet_{\mathfrak{X}}$ of the formal scheme $\mathfrak{X}$ induces a descreasing filtration $F^\bullet$ on the de Rham cohomology groups $H_{DR}(\mathfrak{X})$ which we call the weak Hodge filtration. Of course, the associated spectral sequence does not degenerate. Nevertheless, since $H^p(\mathfrak{X}, \Omega^q_{\mathfrak{X}}) = 0$ when $p > q$, we have $H^k_{DR}(\mathfrak{X}) = F^pH^k_{DR}(\mathfrak{X})$ whenever $k \leq 2p$.

It is well-known that the canonical restriction map

$$H^\bullet_{DR}(\mathfrak{X}) \to H^\bullet(E_y, \mathbb{C})$$

is an isomorphism. By definition (see [D]), to obtain the Hodge filtration on the cohomology groups $H^\bullet(E_y)$, one has to choose a smooth simplicial resolution $E^\bullet_y$ for the variety E_y and take the usual Hodge filtration on $H^\bullet(E^\bullet_y)$. The embedding $E_y \to \mathfrak{X}$ gives a map $E^\bullet_y \to \mathfrak{X}^\bullet$, where $\mathfrak{X}^\bullet$ is $\mathfrak{X}$ considered as a constant simplicial variety. The corresponding restriction map

$$H^\bullet_{DR}(\mathfrak{X}) \to H^\bullet_{DR}(E^\bullet_y)$$

is also an isomorphism, and it sends the weak Hodge filtration on the left-hand side into the usual Hodge filtration on the right-hand side. We conclude that $H^k(E_y) = F^p(E_y)$ whenever $k \leq 2p$. It remains to recall that by definition, we have $H^k(E_y) = W_k H^k(E_y)$, and apply Lemma 3.3. $\square$

To conclude the paper, we would like to note that in the particular case when $X = T^*(G/B)$ is the Springer resolution of the nilpotent cone $Y = \mathcal{N} \subset \mathfrak{g}^*$ in the coadjoint representation $\mathfrak{g}^*$ of a semisimple algebraic group G, Theorem 3.2 has been already proved by C. de Concini, G. Lusztig and C. Procesi in [dCLP]. They proceed by a direct geometric argument. As a result, they obtain more: not only do the cohomology groups carry a Hodge structure of Hodge-Tate type, but in fact they are spanned by cohomology classes of algebraic cycles. This is true even for cohomology groups with integer coefficients. Motivated by this, we propose the following.

Conjecture 3.4. *In the assumptions of Theorem 3.2, the cohomology groups $H^k(E_y, \mathbb{Z})$ are trivial for odd k, and are spanned by cohomology classes of algebraic cycles for even k.*

We also expect that an analogous statement holds for l-adic cohomology groups, possibly even over fields of positive characteristic.

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